ON THE IDEALIZATION OF SURFACE OF CONTACT IN FORM OF POINT CONTACT IN THE PROBLEM OF ROLLING

(OB IDEALIZATSII POVERKHNOSTI SOPRIKOSNOVENIIA V VIDE TOCHECHNOGO KONTAKTA V ZADACHAKH KACHENIIA)

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In many problems of rolling of a rigid body on a surface use is often made of the assumption of point contact [1 to 5]. However, usually the area of contact has a surface which, although small, differs from zero, therefore the introduction of the concept of point contact into mechanics is a specific idealization used to simplify the mathematical model of a real system. In many cases, especially in problems of statics, such idealization may be justified. In dynamics, a different situation may occur, namely: the magnitude of the area of contact, which tends to zero, enters into the differential equation of motion in the form of a small parameter, and then we cannot know in advance whether the final motion of the body will coincide with the motion of the ideal system (i.e. body with point contact) in the limit.

In the present work this question is studied by the example of the motion of an homogeneous sphere on a rough plane. The investigation of the dynamics of the system shows that the motion of a sphere with a surface of contact has a singular quality which remains when the area of contact surface tends to zero. Moreover, it is explained that the trajectory of the center of the sphere when the area of contact tends to zero approaches the trajectory of the center of the sphere with point contact.

1. Formulation of problem and equations of motion. We consider the inertial motion of an homogeneous heavy sphere on a horizontal rough plane. Let the radius a of the area of contact of the sphere with the plane be much smaller than the radius R of the sphere. In this case we may neglect the curvature of the surface of contact, and assume that pressure on the plane of contact is distributed uniformly. We shall introduce a fixed coordinate system $OXYZ$ (Fig. 1) and the following notation: $\mathrm{U}(u, v, 0)$ is velocity of the center of the sphere, ω $(\omega_x, \omega_y, \omega_z)$ is the angular velocity of rotation of the sphere, m its mass, ρ it's radius, of inertia, F $(F_x, F_y, 0)$ is the resultant force of friction, M is the moment of force of friction with respect to the vertical axis passing through the center of the plane of contact.

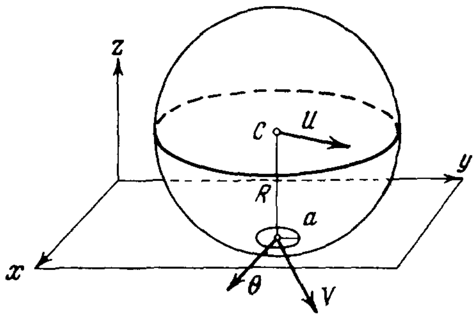

FIG. 1

$V\ (V\cos\theta, V\sin\theta, 0)$ is the velocity of a point on the sphere, coinciding with the center of the plane of contact. In this notation the equations of motion have the form

$$m\frac{du}{dt} = F_x, \qquad m\frac{dv}{dt} = F_y$$
$$m\rho^2\frac{d\omega_x}{dt} = RF_y, \qquad m\rho^2\frac{d\omega_y}{dt} = -RF_x, \qquad m\rho^2\frac{d\omega_z}{dt} = M \tag{1.1}$$

To these equations we add the kinematic relation

$$V\cos\theta = u - R\omega_y, \qquad V\sin\theta = v + R\omega_x \tag{1.2}$$

which in the case of a sphere rolling without slipping ($V = 0$) are changed into nonholonomic equations. From equations (1.1) and (1.2) it directly follows that in the case of motion without slipping the center of the sphere moves in a straight line with constant velocity, as in the case of point contact. We will assume that the forces of interaction of the sphere with the plane are the Coulomb (dry) friction forces. We denote the coefficient of dry friction $\varkappa$. After calculations analogous to these given in the book by Lur'e [6], we obtain

$$F = -\frac{4}{3\pi}mg\varkappa kf_1(k), \qquad M = -\frac{4}{9\pi}mg\varkappa af_3(k) \quad \text{for} \quad V \leqslant a\omega_z$$
$$F = -\frac{4}{3\pi}mg\varkappa f_1(k_1), \qquad M = -\frac{4}{9\pi}mg\varkappa af_2(k_1) \quad \text{for} \quad V \geqslant a\omega_z \tag{1.3}$$

where functions

$$f_1(k) = (k^{-2}+1)E(k) - (k^{-2}-1)K(k) \qquad (k = 1/k_1 = V/a\omega_z)$$
$$f_2(k) = k^{-1}[(4-2k^{-2})E(k) - (k^{-2}-1)(3k^2-2)K(k)] \tag{1.4}$$
$$f_3(k) = (4-2k^2)E(k) - (1-k^2)K(k)$$

are expressed by complete elliptic integrals $K(k)$ and $E(k)$ in terms of the modulus k.

Let us differentiate the relations (1.2) with respect to time and replace the derivatives in their right hand sides with these from (1.1). Putting also $F_x = F\cos\theta$, $F_y = F\sin\theta$ and solving the obtained equations with respect to derivatives, we have

$$\frac{dV}{dt} = \frac{1}{m}\left(1+\frac{R^2}{\rho^2}\right)F, \qquad V\frac{d\theta}{dt} = 0 \tag{1.5}$$

From the second equation of (1.5) it follows that $\theta = \theta_0 =$ const, i.e. the direction of the velocity of slipping of the center of the plane of contact remains fixed. The first equation of (1.5) and the last equation of (1.1) together with (1.3) and (1.4) form a closed system of equations. The study of the motion of a sphere with slipping can be now reduced to the examination of the above system of equations. To simplify the notation, we introduce

$$\Omega = a\omega_z, \qquad \mu = \frac{a^2}{3(R^2+\rho^2)}, \qquad \tau = \frac{4g\varkappa}{3\pi}\left(1+\frac{R^2}{\rho^2}\right)t \tag{1.6}$$

where τ is a new time.

In this notation the obtained equations assume the form

$$V^{\cdot} = -kf_1(k), \qquad \Omega^{\cdot} = -\mu f_3(k) \quad \text{for} \quad V \leqslant \Omega \qquad \left(k = \frac{1}{k_1} = \frac{V}{\Omega}\right) \tag{1.7}$$

$$V^{\cdot} = -f_1(k_1), \qquad \Omega^{\cdot} = -\mu f_2(k_1) \quad \text{for} \quad V \geqslant \Omega \tag{1.8}$$

Here dots will denote differentiation with respect to new time τ; functions f_1, f_2, and f_3 are determined by expressions (1.4). From equations (1.7) and (1.8) it follows that the dynamics of the system considered here, is defined by a unique parameter μ, proportional to the surface area of contact.

2. Motion of a sphere with slipping. In accordance with the system (1.7) and (1.8) the motion of a sphere with slipping may be compared with the motion of a representative point in the first quadrant of the $V\Omega$ plane. This quadrant is divided by the bisector $\Omega = V$ into regions S and S_1; in region S the motion of the point is given by Equation (1.7), and in region S_1 by Equation (1.8). On the boundary $\Omega = V$ the solutions of Equations (1.7) and (1.8) 'are fused' by conditions of continuity. The analysis (cf. section 4 Application) of the behavior of functions f_1, f_2 and f_3 results in the curves shown on Fig. 2. Hence from equations (1.7) and (1.8) it follows that the values of V and Ω are monotone decreasing with time. We shall now consider the qualitative behavior of the integral curves on the V,Ω plane, the equations of which have the form

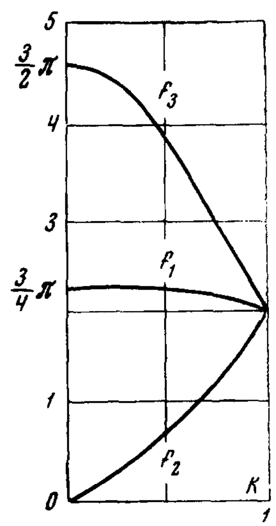

FIG. 2

$$\frac{d\Omega}{dV} = \mu\frac{f_3(k)}{kf_1(k)} \quad \left(k = \frac{V}{\Omega}\right) \qquad (V \leqslant \Omega)$$

$$\frac{d\Omega}{dV} = \mu\frac{f_2(k_1)}{f_1(k_1)} \quad \left(k_1 = \frac{\Omega}{V}\right) \qquad (V \geqslant \Omega) \tag{2.1}$$

The family of isoclines consists of a pencil of straight lines, passing through the origin of coordinates. By Fig. 2, functions entering in the right-hand sides of equations (2.1) and (2.2) are represented by the curves shown on Fig. 3. These curves make it possible to compare the angle of inclination of the tangent to the integral curve with the angle of inclination of the corresponding isocline, and by the same token to construct the vector field of tangents on the $V\Omega$ plane for various values of the parameter μ $(0 \leqslant \mu < \infty)$. For values of μ within the interval $(^1/_2 < \mu < {}^8/_3)$ one of the integral curves is a straight line, passing through the origin of coordinates. Fig. 4, shows the $V\Omega$ plane with the trajectories plotted for small values of μ $(\mu \ll 1)$ and we notice that all the trajectories terminate at the origin of coordinates. The coordinate axes $\Omega = 0$ and $V = 0$ will also be trajectories of motion of the representative point.

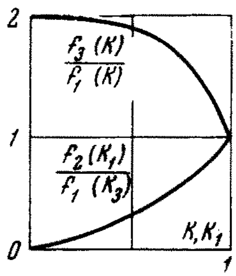

FIG. 3

The above argument leads us to the following conclusion: for any initial condition, for which the velocity V of sliding and the angular velocity ω_z of rotation of the sphere differ from zero, the sphere with a surface contact moves so that the rotation and sliding diminish with time and always cease simultaneously.

3. Limiting motion of a sphere when surface area of contact tends to zero. The motion of a sphere with point contact (limiting system) is described by the equations (1.8) in which transition to the limit $a \to 0$ must be made. Then, from the second of the equations (1.8) we obtain $\omega_z = \text{const}$, while the first equation yields $V = V_0 - {}^3/_4\pi\tau$. On the plane (V, Ω) the representative point moves along the axis $\Omega = 0$, reaching the origin of coordinates after a finite interval of time $T^\circ = {}^4/_3 V_0 / \pi$. At the same time the center of the sphere describes a parabola, while after the time T° it begins to move a straight line.

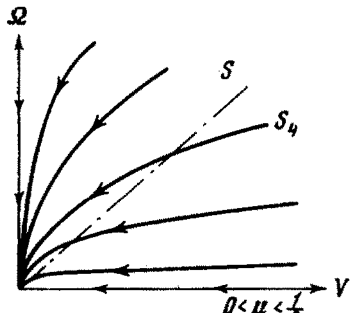

FIG. 4

We shall consider the motion of a sphere with its area of contact with the plane of contact tending to zero. The motion in this case is described by the system of Equations (1.7) and (1.8) in which $0 < \mu \ll 1$. Let us express this system in one Equation

$$d\Omega / dV = \mu\Phi(k) \qquad (k = V / \Omega) \tag{3.1}$$

The function $\Phi(k)$, as follows from Fig. 3 and the right-hand sides of Equations (2.1) and (2.2), is represented by a curve contained between hyperbolas $3/8k$ and $2/k$. Therefore [7] the solution of Equation (3.1) lies between the solutions

$$\begin{aligned} \Omega_1 &= \Omega_0 (V / V_0)^{3/8\mu} \\ \Omega_2 &= \Omega_0 (V / V_0)^{2\mu} \end{aligned} \tag{3.2}$$

Here Ω_0 and V_0 are the initial values. From Fig. 4 it follows that for any initial values, for which $\Omega_0 \neq 0$, the representative point will always eventually arrive to the region S. Let us estimate the time T_1 of the motion of the representative point in region S, if, in the initial moment, the point lies on the boundary of the regions S and S_1. In this case $V_0 = \Omega_0$, and the solutions (3.2) are written in the form

$$\Omega_1 = \Omega_0^{(1-{}^3/_8\mu)} V^{{}^3/_8} \mu, \quad \Omega_2 = \Omega_0^{(1-2\mu)} V^{2\mu} \tag{3.3}$$

The first equation of (1.7) reaches its maxima at $V^{\cdot} = -{}^3/_4\pi k$ and $V^{\cdot} = -2k$. Substituting here the solutions (3.3), we obtain

$$V^{\cdot} = -ACV^{(1-B)} \tag{3.4}$$

Here the constants A, B and C are correspondingly equal to one of two values

$$A = 2,\ {}^3/_4\pi; \qquad B = {}^3/_8\mu,\ 2\mu; \qquad C = \Omega_0^{({}^3/_8\mu-1)},\ \Omega_0^{(2\mu-1)} \tag{3.5}$$

The solution of Equation (3.4) has the form

$$V^B = \Omega_0^B - ABC\tau \tag{3.6}$$

Assuming here $V = 0$, we find the time $\tau = T_1$ of transition of the representative point from the boundary $\Omega_0 = V_0$ to the origin of coordinates. According to (3.5) the time T_1 lies within the interval

$$\frac{2\Omega_0^{(1-{}^{13}/_8\mu)}}{3\pi\mu} < T_1 < \frac{4\Omega_0^{(1+{}^{13}/_8\mu)}}{3\pi\mu} \tag{3.7}$$

Hence, by (1.6), it follows that limit $T_1 = \infty$ for $a \to 0$. Thus for an arbitrarily small but non-zero area of contact the time of motion of the sphere with slipping becomes arbitrarily large.

This illustrates the qualitative difference between the motion of a sphere with a surface of contact vanishingly small, and the motion of a sphere with point contact.

For the path length s of the motion of representative point in the region S we have

$$s = \int_0^{T_1} V\,d\tau = -\int_{V_0}^{0} \frac{V\,dV}{ACV^{(1-B)}} = \frac{1}{AC}\int_0^{\Omega} V^B\,dB = \frac{\Omega_0^{(1-B)}}{AC\,(1+B)} \tag{3.8}$$

Here for $d\tau$ we have used the expression (3.4). From the equality (3.8) it follows that $\lim s = 0$ as $a \to 0$, i.e. for $a \to 0$ the trajectory of the center of the sphere with a surface contact approaches the trajectory of a sphere with point contact.

4. Application. We again note that on the interval $0 \leqslant x \leqslant 1$ the function $K(x)$ represents a curve monotonely increasing from the value $K(0) = {}^1/_2\pi$ to infinity. The derivative

$$\frac{dK}{dx} = \frac{1}{x}\left[\frac{1}{1-x^2}E - K\right] > 0$$

increases monotonely from zero to infinity. The function $E(x)$ represents a curve monotonely decreasing from $E(0) = {}^1/_2\,\pi$ to $E(1) = 1$. The derivative

$$\frac{dE}{dx} = \frac{1}{x}(E - K) < 0$$

monotonely decreases from zero to $-\infty$.

From (1.4) it follows that $f_1(1) = f_2(1) = f_3(1) = 2$. Also

$$f_1(0) = \frac{3}{4}\pi, \qquad f_1'(x) = -\frac{2}{x^2}\left[\left(1-\frac{x^2}{2}\right)E - (1-x^2)K\right] > 0$$

since

$$\varphi(x) \equiv (1 - {}^1/_2\,x^2)\,E - (1 - x^2)\,K > 0$$

Indeed

$$\varphi(0) = 0, \qquad \varphi'(x) = {}^3/_2\,x\,(K - E) > 0$$

Calculating the second derivative, we find

$$f_1''(x) = -\frac{6}{x^4}\left[\left(1-\frac{x^2}{2}\right)K - E\right] < 0$$

In fact, the function $\varphi_1(x) \equiv (1 - {}^1/_2\,x^2)\,K - E$ is positive everywhere on the interval $0 \leqslant x \leqslant 1$ since

$$\varphi_1(0) = 0, \qquad \varphi_1'(x) = \frac{x}{2}\left(\frac{1}{1-x^2}E - K\right) > 0$$

Thus the function $f_1(x)$ will be monotonely decreasing, and its first derivative will decrease from the value $f_1'(0) = 0$ monotonely to the value $f_1'(1) = -1$.

Let us now investigate the behavior of the function $f_2(x)$. We find

$$f_2(0) = 0, \qquad f_2'(x) = \frac{6}{x^4}\left[\left(1-\frac{x^2}{2}\right)E - (1-x^2)K\right] > 0$$

since the function $\varphi_2(x) \equiv (1 - {}^1/_2\,x^2)\,E - (1 - x^2)\,K$ is positive. Indeed,

$$\varphi_2(0) = 0, \qquad \varphi_2'(x) = {}^3/_2\,x\,(K - E) > 0$$

The second derivative is also positive

$$f_2''(x) = \frac{3}{x^5}\left[(8 - 3x^2)K - (8 - x^2)E\right] > 0$$

since the function $\varphi_3(x) \equiv (8 - 3x^2)\,K - (8 - x^2)\,E$ is positive.

Indeed,

$$\varphi_3(0) = 0, \qquad \varphi_3'(x) = x\left(\frac{8-3x^2}{1-x^2}E - 4K\right) > 0$$

because the expression in parentheses is the sum of two positive terms

$$\frac{8-3x^2}{1-x^2}E - 4K = 4\left(\frac{1}{1-x^2}E - K\right) + \frac{4-3x^2}{1-x^2}E$$

Therefore the function $f_2(x)$ is monotonely increasing from the value $f_2(0) = 0$ to the value $f_2(1) = 2$. The first derivative also increases monotonely from $f_2'(0) = {}^9/_{32}\pi$ to $f_2'(1) = 3$.

Finally, let us investigate the behavior of function $f_3(x)$. We find

$$f_3(0) = \frac{3\pi}{2}, \qquad f_3'(x) = -\frac{3}{x}[(1-x^2)K - (1-2x^2)E) < 0$$

since the function $(1-x^2)K - (1-2x^2)E = (1-x^2)(K-E) + x^2E$ is positive.

Further,

$$f_3''(x) = \frac{3}{x}[(1+2x^2)K - (1+4x^2)E], \qquad f_3''(0) = -\frac{9\pi}{4} < 0, \quad f_3''(1) = +\infty$$

The function $f_3''(x)$ becomes equal to zero only once, since

$$f_3'''(x) = \frac{3}{x^4}\left(\frac{1}{1-x^2}E - K\right) > 0$$

Therefore, the curve $f_3 = f_3(x)$ emerges from the point $(0, {}^3/_2\pi)$ with the angle of inclination of its tangent equal to zero. The angle of inclination then diminishes until, the point $x = x^\circ$, where x° is a root of the equation $(1+2x^2)K = (1+4x^2)E$ after which it increases to the value for which the derivative is equal to $f_3'(1) = -3$.

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